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Minimally 3-connected binary matroids

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Abstract

A 3-connected matroid M is said to be *minimally 3-connected* if, for any element e of M, the matroid $M\backslash e$ is not 3-connected. Dawes [R.W. Dawes, Minimally 3-connected graphs, J. Combin. Theory Ser. B 40 (1986) 159–168] showed that all minimally 3-connected graphs can be constructed from K_4 such that every graph in each intermediate step is also minimally 3-connected. Oxley [J.G. Oxley, On connectivity in matroids and graphs, Trans. Amer. Math. Soc. 265 (1981) 47–58] proved a similar result by giving a characterization of minimally 2-connected matroids. In this paper we generalize Dawes' result to minimally 3-connected binary matroids. We give a constructive characterization of all minimally 3-connected binary matroids starting from $\mathcal{W}_3$, the 3-spoked wheel, and F_7^*, the Fano dual.

1. Introduction

The terminology used here for graphs and matroids will follow Oxley [7]. A k-connected graph G, is said to be minimally k-connected if, for any edge e of G, the graph $G\backslash e$ is not k-connected. Similarly, a k-connected matroid M is *minimally k-connected*, if, for any element e of M, we have that $M\backslash e$ is not k-connected. Dually, we define a k-connected matroid M to be *cominimally k-connected* if, for any element e of M, the matroid M/e is not k-connected. There are many results on 3-connected matriods in the literature, see [1,2,4–11,13], for example. The following is the celebrated Wheels and Whirls Theorem of Tutte [11].

Theorem 1. *Let M be a 3-connected matroid. Then either*

1. *There is an element e of M, such that the deletion $M\backslash e$ of e from M is 3-connected, or*
2. *There is an element e of M, such that the contraction M/e of e from M is 3-connected, or*
3. *M has rank at least three and is isomorphic to a wheel or a whirl.*

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We will use $\lceil M \rceil$ to denote the simple matroid associated with M and $\lfloor M \rfloor$ the cosimple matroid associated with M. A *triad* in a matroid is a 3-element cocircuit. A *k-circuit* is a k-element circuit, and a *k-cocircuit* is a k-element cocircuit. A *triad* in a 3-connected graph is the set of edges incident to the same vertex of degree three. The corresponding concept in a matroid is a vertex-triad [13]. A triad T^* in a matroid M is a *vertex-triad* if $M \backslash T^*$ is connected. A *contractible triangle* T in a matroid M is a triangle such that M/T is connected. Clearly, T is a contractible triangle in M if and only if it is a vertex-triad in the dual matroid M^*.

When G is a minimally 3-connected graph, Tutte's Wheels Theorem shows that G can be constructed from a wheel. However, the graphs in the intermediate steps of this construction are not necessarily minimally 3-connected. As a result, one has to go out of the class of minimally 3-connected graphs to construct the class of minimally 3-connected graphs. This is often undesirable. The following result of Dawes [3] states that every minimally 3-connected graph can be constructed from K_4 such that the graph obtained at each step is minimally 3-connected. It gives a constructive characterization of minimally 3-connected graphs. Here $\lfloor G \backslash e \rfloor$ (in the next theorem) is obtained from $G \backslash e$ by contracting all but one edge in each series class.

Theorem 2. *Let G be a minimally 3-connected graph on at least four vertices. Then, either*
1. *G has a triad T^* and $e \in T^*$ such that $\lfloor G \backslash e \rfloor$ is minimally 3-connected, or*
2. *G has a degree-three vertex u such that $G \backslash u$ is minimally 3-connected, or*
3. *$G = K_4$.*

This result provides a useful tool for induction arguments. By starting with a minimally 3-connected graph G, a smaller minimally 3-connected graph can be obtained. It is natural to seek generalizations of this result to 3-connected matroids. The corresponding problem for connected matroids has been solved by Oxley [5].

Theorem 3. *A matroid M is minimally connected if and only if $|E(M)| \geq 3$, and either M is connected and has every element in a 2-cocircuit, or $M = S((M_1/q_1; p_1), (M_2/q_2; p_2))$ where both M_1 and M_2 are minimally connected matroids having at least five elements, and $\{p_1, q_1\}$ and $\{p_2, q_2\}$ are cocircuits of M_1 and M_2, respectively.*

In this paper, we will give a generalization of Theorem 2 to minimally 3-connected binary matroids. We shall use $\mathcal{W}_r$ to denote the rank-r wheel. The following theorem is the main result of the paper.

Theorem 4. *Let M be a minimally 3-connected binary matroid on at least four elements. Then, either*
1. *M has a vertex-triad T^* and $e \in T^*$ such that $\lfloor M \backslash e \rfloor$ is minimally 3-connected, or*
2. *M has a vertex-triad T^* such that $M \backslash T^*$ is minimally 3-connected, or*
3. *$M = \mathcal{W}_3$ or F_7^*.*

It is clear that Theorem 2 is an immediate consequence of Theorem 4. Oxley's theorem gives a reduction of a minimally connected matroid M to two smaller minimally connected matroids M_1 and M_2, which is very useful in induction arguments (see [5]). It also gives a characterization for minimally connected matroids using series-connection. Our main result is a natural extension of both Theorems 2 and 3 and should be useful for applications where minimal 3-connectivity must be maintained. In Section 2, we give some preliminary results needed for our proof. The proof of our main result will be delayed until Section 3. In Section 4, we will give an example which shows that our theorem cannot be extended to general 3-connected matroids.

2. Preliminaries

The ground set and rank of a matroid M will be denoted by $E(M)$ and $r(M)$ respectively. The series and parallel connections of matroids M_1 and M_2 will be denoted $S(M_1, M_2)$ and $P(M_1, M_2)$, respectively. Let M and N be matroids each with at least two elements. Let $E(M) \cap E(N) = \{p\}$ and suppose that neither M nor N has $\{p\}$ as a separator. Then the *2-sum* $M \oplus_2 N$ of M and N is $S(M, N)/p$ or, equivalently, $P(M, N)\backslash p$. The element p is called the *basepoint* of the 2-sum, and M and N are the *parts* of the 2-sum. For convenience $M/\{f, g\}$ and $M\backslash\{f, g\}$ will be abbreviated as $M/f, g$ and $M\backslash f, g$, respectively.

Let M be a matroid and k be a positive integer. A *k-separation* of M is a partition $\{X, Y\}$ of $E(M)$ such that

$$\min\{|X|, |Y|\} \geq k \tag{1}$$

and

$$r(X) + r(Y) - r(M) \leq k - 1. \tag{2}$$

A matroid M is *k-separable* if it has a k-separation. For all $n \geq 2$, M is n-connected if, for all k in $\{1, 2, \ldots, n-1\}$, M has no k-separation. A matroid M is 2-connected if and only if M is connected. If M has no k-separation for all $k > 0$, then we say that M has infinite connectivity. For a positive integer k, a *vertical k-separation* of M is a partition $\{X, Y\}$ of $E(M)$ that satisfies (2) and the following strengthened form of (1):

$$\min\{r(X), r(Y)\} \geq k.$$

Let M be a 3-connected matroid. An element e is *deletable* if $M\backslash e$ is 3-connected and e is *contractible* if M/e is 3-connected. An element e in a 3-connected matroid M is *essential* if neither M/e nor $M\backslash e$ is 3-connected.

Let $F = \{x_1, x_2, \ldots, x_k\}$ be a subset of a matroid. Then F is called a *fan* if for all $i = 1, 2, \ldots, k-2$, the following are true:

(i) $\{x_i, x_{i+1}, x_{i+2}\}$ is a triangle or triad, and

(ii) If $\{x_i, x_{i+1}, x_{i+2}\}$ is a triangle, then $\{x_{i+1}, x_{i+2}, x_{i+3}\}$ is a triad; while $\{x_i, x_{i+1}, x_{i+2}\}$ is a triad, then $\{x_{i+1}, x_{i+2}, x_{i+3}\}$ is a triangle.

If F is a maximal fan, then x_1 and x_k are called the *ends* of the fan. If each end is in some triangle, but not in any triad, then F is called a *type*-1 maximal fan. If each end is in a triad, but not in any triangle, then F is called a *type*-2 maximal fan. Otherwise, F is called a *type*-3 maximal fan. The following result shows that an essential element in a 3-connected matroid other than a wheel or a wheel is in a maximal fan; both ends of the fan are non-essential; and this maximal fan is unique except in some special cases.

Theorem 5 (*[9]*). *Let M be a* 3*-connected matroid that is not a wheel or a whirl. Suppose that e is an essential element of M. Then e is in a maximal fan, both ends of which are non-essential. Moreover, this maximal fan is unique unless*

(i) *every maximal fan containing e consists of a single triangle and any two such triangles meet in $\{e\}$; or*

(ii) *every maximal fan containing e consists of a single triad and any two such triads meet in $\{e\}$; or*

(iii) *e is in exactly three maximal fans; these three fans are of the same type; each has five elements. Together they contain a total of six elements; and, depending on whether these fans are of type-1 or type-2, the restriction or contraction, respectively, of M to this set of six elements is isomorphic to $M(K_4)$.*

The following three results can be found in Oxley [7].

Lemma 6 (*[7, Proposition 7.1.15]*). *Let M_1 and M_2 be matroids with $E(M_1) \cap E(M_2) = \{p\}$. If $e \in E(M_1) - p$, then*
$S(M_1, M_2)/e = S(M_1/e, M_2)$;
$S(M_1, M_2)\backslash e = S(M_1\backslash e, M_2)$;
$P(M_1, M_2)/e = P(M_1/e, M_2)$; *and*
$P(M_1, M_2)\backslash e = P(M_1\backslash e, M_2)$. □

Lemma 7 (*[7, Proposition 7.1.20]*). *Let M and N be two matroids. Then the following are true:*
1. $(M \oplus_2 N)^* = M^* \oplus_2 N^*$.
2. *If $|E(M)|, |E(N)| \geq 2$, then $M \oplus_2 N$ is connected if and only if both M and N are connected.* □

Lemma 8 (*[7, Proposition 8.1.12]*). *Let (X, Y) be a k-separation of a matroid M and suppose that $|Y| \geq k + 1$. Then either X is both a flat and a coflat of M, or for some element e of Y, $(X \cup e, Y - e)$ is a k-separation of M.* □

Lemma 9. *Let M be a simple binary matroid with at least two elements. Suppose that M is not 3-connected, but $N = \lceil M/e \rceil$ is 3-connected. Then e is a coloop or in a 2-element cocircuit of M.*

Proof. Suppose that M is not connected. Then $M = M_1 \oplus M_2$ for two submatroids M_1 and M_2 of M. Suppose that $e \in E(M_1)$. As $\lceil M/e \rceil$ is 3-connected, $\lceil M_1/e \rceil$ must be the empty matroid. Thus we deduce that $E(M_1) = \{e\}$ as M is simple. Therefore e is a coloop of M.

Now we assume that M is connected but not 3-connected. Assume that e is not in any 2-element cocircuit of M. Choose a 2-separation (X, Y) of M such that $e \in X$ and with this condition, $|X|$ is minimum. Clearly $|X| \geq 3$ as M is simple. As M is not 3-connected, there are two connected matroids M_1 and M_2 such that $M = M_1 \oplus_2 M_2$, where $E(M_1) = X \cup p$ and $E(M_2) = Y \cup p$ for some new element p. As $M/e = (M_1/e) \oplus_2 M_2$ and $N = \lceil M/e \rceil$ is 3-connected, we deduce that $r(M_1/e) = 1$. Hence $r(M_1) = 2$. As M is simple and binary and $|X| \geq 3$, we conclude that M_1 has exactly four elements and $M_1\backslash p \cong U_{2,3}$. Moreover, p is parallel to some element x in M_1. As $\lceil M/x \rceil$ has a coloop and thus is not 3-connected, we conclude that $e \neq x$. Now it is clear that $X - x$ is a 2-element cocircuit containing e, a contradiction. This completes the proof of the lemma. □

The proof of the next lemma is straightforward and will be omitted.

Lemma 10. *Let e be an element of a matroid M that is not in any circuit of size less than three. Then $\lceil M/e \rceil \cong \lceil M \rceil/e$ if and only if e is not in any triangle of M.*

Lemma 11. *Let M be a 3-connected binary matroid other than F_7 or $\mathcal{W}_3$ and assume that M has at least four elements. Suppose that M/f is 2-separable for some element $f \in E(M)$. Suppose that $g \in E(M)$. If $\lceil M/g \rceil/f$ is 3-connected, then f is in a unique triangle of M, and this triangle contains g.*

Proof. Suppose that $\{f, g\}$ is not in any triangle of M. First, suppose that f is in some triangle of M. As $\{f, g\}$ is not in any triangle of M, we conclude that $\lceil M/g \rceil$ has a triangle containing f. Hence $\lceil M/g \rceil / f$ has a pair of parallel elements and thus is not 3-connected unless $\lceil M/g \rceil / f \cong U_{1,2}, U_{1,3}$. Thus $r(M/g) = 2$ and $r(M) = 3$. As M is binary, M is a submatroid of F_7. Therefore as M is 3-connected and has at least four elements, we deduce that $M \cong F_7$ or $M \cong \mathcal{W}_3$, a contradiction.

Therefore f is not in any triangle of M. Then M/f is simple but not 3-connected. Suppose that $\lceil M/g \rceil / f$ is not simple. Then as $\lceil M/g \rceil / f$ is 3-connected, we deduce that $r(\lceil M/g \rceil / f) = 1$. Thus $r(M) = 3$. We conclude that $M \cong F_7$ or $\mathcal{W}_3$, a contradiction. Therefore $\lceil M/g \rceil / f$ is simple. As M is 3-connected and has at least four elements, each circuit has size at least three. Thus f is not in any circuit of size less than three in M/g as f is not in any triangle of M. Next we show that f is not in any triangle of M/g. Suppose not and T is such a triangle. Then T is also a triangle of $\lceil M/g \rceil$. Hence $\lceil M/g \rceil / f$ is not simple, a contradiction. By Lemma 10, we have $\lceil M/g/f \rceil \cong \lceil M/g \rceil / f$. Therefore $\lceil M/f/g \rceil = \lceil M/g/f \rceil \cong \lceil M/g \rceil / f$ is 3-connected, but M/f is simple and not 3-connected. By Lemma 9, taking the matroid M/f as M, we have that g is a coloop or in a 2-element cocircuit of M/f. We conclude that g is a coloop or in a 2-element cocircuit of M, a contradiction as M is 3-connected with at least four elements.

We conclude that $\{f, g\}$ is in a triangle. Now as M is binary and simple, this triangle is unique. Suppose that f is in more than one triangle. As M is binary, each line containing f has exactly three elements. Hence $\lceil M/g \rceil$ has at least one triangle containing f. Thus as $\lceil M/g \rceil / f$ is 3-connected, we have that $\lceil M/g \rceil / f \cong U_{1,2}$ or $U_{1,3}$. This implies that $r(M) = 3$. Hence $M \cong F_7$ or $\mathcal{W}_3$, a contradiction. □

3. Proofs

In this section we will give a proof of the main result. The following is the dual statement of Theorem 4.

Theorem 12. *Let M be a cominimally 3-connected binary matroid on at least four elements. Then, either*

1. *M has a contractible triangle T and $e \in T$ such that $\lceil M/e \rceil$ is cominimally 3-connected, or*
2. *M has a contractible triangle T such that M/T is cominimally 3-connected, or*
3. *$M = \mathcal{W}_3$, or F_7.*

We will need the following lemmas to prove this theorem.

Lemma 13. *Let M be a cominimally 3-connected matroid with at least four elements. Then M has an element e contained in some triangle such that $\lceil M/e \rceil$ is 3-connected.*

Proof. Cunningham [2], and independently, Seymour [10] showed that M has an element e such that $\lceil M/e \rceil$ is 3-connected. As M is cominimally 3-connected, we conclude that M/e is not 3-connected. Hence $M/e \neq \lceil M/e \rceil$. Therefore M/e is not simple. As each circuit of M has at least three elements, we conclude that e belongs to some triangle. □

Lemma 14. *Let M be a cominimally 3-connected binary matroid other than F_7 and assume that $|E(M)| \geq 4$. Suppose that M has an element e that is in a triangle of M such that $\lceil M/e \rceil$ is 3-connected. Then, either M has a 4-element fan, or a triangle T such that M/T is cominimally 3-connected, or an element f that is in a triangle of M and for which $\lceil M/f \rceil$ is cominimally 3-connected.*

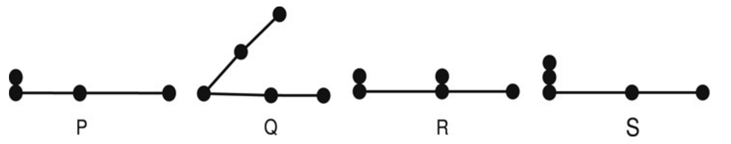

Fig. 1. P, Q, R, and S.

Proof. Assume that M does not have a 4-element fan. Then M is not isomorphic to $\mathcal{W}_3$, which has a 4-element fan. If $\lceil M/e \rceil$ is cominimally 3-connected, then we are done (by letting $f = e$). So, suppose that $\lceil M/e \rceil$ is not cominimally 3-connected. Then there exists an element f such that $\lceil M/e \rceil/f$ is 3-connected. As M/f is not 3-connected, by Lemma 11, we deduce that $\{e, f\}$ is contained in a unique triangle $T = \{e, f, g\}$ for some $g \in E(M)$. Moreover, T is the unique triangle containing f.

Case 1. M/T is 3-connected. If M/T is cominimally 3-connected, then we are done. So suppose it is not. Then there exists an element $k \in E(M/T)$ such that $M/T/k$ is 3-connected. As M/k is not 3-connected, it has a 2-separation (X, Y). Then $M/k = M_1 \oplus_2 M_2$ for some matroids M_1 and M_2, where M_1 and M_2 are isomorphic to proper minors of M/k. Moreover, $|E(M_1)|, |E(M_2)| \geq 3$; $E(M_1) = X \cup p$ and $E(M_2) = Y \cup p$; and p is the basepoint of the 2-sum. By Lemma 7, we have that both M_1 and M_2 are connected.

Without loss of generality, assume that $|X \cap T| \geq 2$. First suppose that $|Y| = 2$. Then, by Oxley [7, Corollary 8.1.11], we have that Y is either a coindependent circuit or an independent cocircuit of M/k. The latter cannot happen as M has no 2-element cocircuit. Hence Y is a 2-element circuit of M/k and thus $Y \cup k$ is a triangle of M. If $T \cap Y \neq \emptyset$, then T and Y have exactly one common element as $|X \cap T| \geq 2$. But then $M/T/k$ has a loop as $Y \cup k$ is a triangle of M. As $M/T/k$ is 3-connected, we deduce that $M/T/k \cong U_{0,1}$. Thus $r(M) = 3$ and $M \cong F_7$ or $\mathcal{W}_3$ as M is 3-connected and binary. This contradiction shows that $T \subseteq E(M_1)$ when $|Y| = 2$. When $|Y| \geq 3$, we will show that we may assume that $T \subseteq E(M_1)$. If $|X \cap T| = 2$, say, $t \in T \backslash X$, then by Lemma 8, we have that $(X \cup t, Y - t)$ is also a 2-separation of M/k as $|Y| \geq 3$.

By Lemma 6, we have that $M/T/k = (M_1/T) \oplus_2 M_2$. As $M/T/k$ is 3-connected, we conclude that M_1/T has at most two elements and is connected. Hence M_1 has either four or five elements and has a triangle T. As M_1 is binary and connected, it is isomorphic to either P, Q, R, or S, which are geometrically represented in Fig. 1.

If M_1 is isomorphic to R or S, then M_1/T is not connected, a contradiction. If M_1 is isomorphic to P or Q, then M_1 contains a 2-element cocircuit avoiding the basepoint p of the 2-sum (note that p is not in the triangle T). This implies that M/k has a 2-element cocircuit and hence M has a 2-element cocircuit, a contradiction. Therefore M/T is cominimally 3-connected.

Case 2. Suppose M/T is not 3-connected. We know that $\lceil M/e \rceil/f$ is 3-connected. Moreover, we know that f and $\{e, f\}$ are in the unique triangle T. Hence $N = M/f \backslash g$ is simple. We shall now show that $N \cong \lceil M/f \rceil$ is 3-connected. Suppose that N is not 3-connected. Suppose e is in exactly $k + 1$ triangles $T, T_1, \ldots, T_k$ for some $k \geq 0$, where $T_k = \{e, f_k, g_k\}$, see Fig. 2.

Claim $\lceil M/e \rceil/f \cong N/e \backslash g_1, g_2, \ldots, g_k \cong \lceil N/e \rceil$.

In fact, $\lceil M/e \rceil/f \cong M/e \backslash g, g_1, \ldots, g_k/f = (M/f \backslash g)/e \backslash g_1, g_2, \ldots, g_k = N/e \backslash g_1, g_2, \ldots, g_k$. Hence the first equality of the claim holds. Now we show that $T_1, T_2, \ldots, T_k$ are the only triangles of $N = M/f \backslash g$ containing e. Suppose not and H is a triangle of N containing e, where $H \neq T_1, T_2, \ldots, T_k$. Then H is also a triangle of M/f. Hence either H or $H \cup f$ is a circuit of M. If the former occurs, then $H = T$ as e is in exactly $k+1$ triangles and $H \neq T_1, T_2, \ldots, T_k$. This is a contradiction as $f \notin H$. Hence $H \cup f$ is a circuit of M. We conclude that $(H - e) \cup f$ is

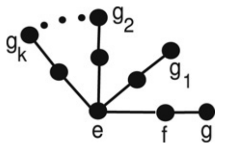

Fig. 2. Triangles containing e.

a triangle of both M/e and $\lceil M/e \rceil$. But then $\lceil M/e \rceil/f$ has a 2-element circuit and we deduce that $\lceil M/e \rceil/f \cong U_{1,2}$ or $U_{1,3}$. We conclude that $r(M) = 3$ and that $M \cong \mathcal{W}_3$ or F_7, a contraction. Therefore $T_1, T_2, \ldots, T_k$ are the only triangles of $N = M/f\backslash g$ containing e. We deduce that $N/e\backslash g_1, g_2, \ldots, g_k \cong \lceil N/e \rceil$. This completes the proof of the claim. □

Now N is simple, not 3-connected, but $\lceil N/e \rceil$ is 3-connected by the above claim. By Lemma 9, e is a coloop or is in a 2-element cocircuit of $M/f\backslash g$. This implies that M has a 2-element cocircuit, or there is a triad T^* of M, such that $\{e, g\} \subset T^*$. As M is 3-connected with at least four elements, M cannot have a 2-element cocircuit. Hence M has a triad containing both e and g. We conclude that M has a 4-element fan, a contradiction. Therefore $\lceil M/f \rceil \cong N$ is 3-connected. If $\lceil M/f \rceil$ is cominimally 3-connected, then we are done. Suppose it is not cominimally 3-connected. Then there exists an element h of M such that $\lceil M/f \rceil/h$ is 3-connected. As M/h is not 3-connected, by Lemma 11, we have that $\{f, h\}$ is in a triangle $T' = \{x, f, h\}$ of M for some element x of M. Since f is contained in a unique triangle of M, we deduce that $T' = T$ and $\{x, h\} = \{e, g\}$. But then $M/T = M/f, g, e = M/f, g\backslash e = M/f\backslash e/g \cong \lceil M/f \rceil/g$ is 3-connected, a contradiction. Therefore $\lceil M/f \rceil$ is cominimally 3-connected. This completes the proof of the lemma. □

Lemma 15. *Let M be a cominimally 3-connected binary matroid other than $\mathcal{W}_3$ and F_7 having a 4-element fan. Then M has an element x in a triangle such that $\lceil M/x \rceil$ is cominimally 3-connected.*

Proof. If $M \cong \mathcal{W}_n$ for some $n \geq 4$, then it is easy to verify that the lemma holds. Thus we will assume that $M \not\cong \mathcal{W}_n$ for any n. Let e be an element of the 4-element fan which is in both a triangle and a triad. Then M/e has a pair of parallel elements. If M/e is 3-connected, then M/e has rank one. Thus $r(M) = 2$. As M is binary, $M \cong U_{2,3}$, a contradiction as M has at least four elements. Thus M/e is not 3-connected. The matroid $M\backslash e$ has a pair of elements in series. If $M\backslash e$ is 3-connected, then $M\backslash e \cong U_{2,3}$ and hence M has exactly four elements. Thus $M \cong U_{2,4}$, a contradiction. We conclude that e is essential. By Theorem 5, there is a maximal fan $\mathcal{F}$ containing the 4-element fan. As M is cominimally 3-connected, both ends of $\mathcal{F}$, denoted by a and b, are both deletable but not contractible (see Fig. 3). As $\mathcal{F}$ contains the 4-element fan, it has at least five elements.

Let f be a rim element of $\mathcal{F}$ such that $\{a, f\}$ is in the same triangle T of $\mathcal{F}$. Now $\lfloor M\backslash f \rfloor$ has a pair of parallel elements and has rank at least two. Hence it is not 3-connected. Therefore $\lceil M/f \rceil$ is 3-connected by Bixby [1]. If $\lceil M/f \rceil$ is cominimally 3-connected, we are done. Otherwise there is an element h in $\lceil M/f \rceil$ such that $\lceil M/f \rceil/h$ is 3-connected. As M is cominimally 3-connected, M/h is not 3-connected. By Lemma 11, we deduce that $\{f, h\}$ is in some unique triangle, denoted by T_1, which is also the unique triangle containing h. Now we show that $\mathcal{F}$ is the unique maximal fan containing f. Suppose not. Then by Theorem 5, the element f is in exactly three 5-element maximal fans and these fans contain a total of six elements. Moreover, the restriction of M to this set of six elements is isomorphic to $M(K_4)$. Now T_1 belongs to a fan

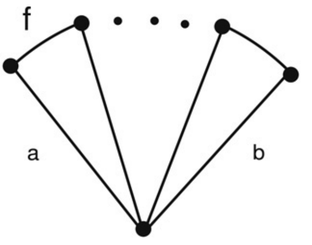

Fig. 3. A fan $\mathcal{F}$.

containing both f and h. Hence h is contained in one of these three fans. We conclude that h is in at least two triangles, a contradiction. Therefore, f is in the unique maximal fan $\mathcal{F}$. We deduce that T is the only triangle of M containing f and hence $T_1 = T$. Thus $h = a$ as h belongs to precisely one triangle. It follows that h is in a triangle of $\lceil M/f \rceil$ as $\mathcal{F}$ has at least five elements. We conclude that $\lceil M/f \rceil/h$ has a 2-element circuit. As $\lceil M/f \rceil/h$ is 3-connected, we deduce that $\lceil M/f \rceil/h$ has at most three elements and $r(\lceil M/f \rceil/h) \leq 1$. Therefore $r(M) \leq 3$. As M is 3-connected and binary, we conclude that $M \cong \mathcal{W}_3$ or F_7. This contradiction completes the proof of the lemma. $\square$

Next we will prove Theorem 12.

Proof. Suppose that M is a cominimally 3-connected matroid with at least four elements and that $M \ncong \mathcal{W}_3, F_7$. By Lemma 13, M has an element t belonging to a triangle such that $\lceil M/t \rceil$ is 3-connected. Thus by Lemma 14, either M has a 4-element fan, or M has a triangle T and an element $e \in T$ such that $\lceil M/e \rceil$ is cominimally 3-connected, or the second statement of the theorem holds. In the case when M has a 4-element fan, by Lemma 15, M has an element x in a triangle such that $\lceil M/x \rceil$ is cominimally 3-connected.

Suppose that e is in a triangle $T = \{e, f, g\}$ of M such that $\lceil M/e \rceil$ is cominimally 3-connected. It suffices to show that T is a contractible triangle. Suppose e is in exactly $k + 1$ triangles $T, T_1, \ldots, T_k$ for some $k \geq 0$, where $T_k = \{e, f_k, g_k\}$; see Fig. 2. Then as $M/e\backslash g, g_1, g_2, \ldots, g_k \cong \lceil M/e \rceil$ is 3-connected, $\lceil M/e\backslash g \rceil$ is also 3-connected. Hence the simple matroid associated with $M/e\backslash g/f$ is connected. Moreover, the last matroid does not have any loops. Indeed, suppose that $M/e\backslash g/f$ would contain a loop, say, p (which is different from g). Then as M is 3-connected matroid with at least four elements, each circuit of M must have size at least three. Hence $\{e, f, p\}$ must be a triangle. Therefore $M|\{e, f, g, p\} \cong U_{2,4}$, a contradiction. We conclude that $M/T = M/e, f, g = M/e\backslash g/f$ is connected. This completes the proof of the theorem. $\square$

4. Further remarks

In this section, we will show that our main result cannot be generalized to general 3-connected matroids. We will also give a counterexample of a conjecture of Wagner [12].

The matroid H_{10} shown in Fig. 4 is cominimally 3-connected as each element is in a triangle. For any element e, we have that $\lceil H_{10}/e \rceil = U_{2,4}$, which is not cominimally 3-connected. Moreover, for any triangle T of M, we have that $H_{10}/T \cong U_{1,7}$, or $U_{1,6} \oplus U_{0,1}$, which are not cominimally 3-connected. Therefore, the assumption that M is binary in Theorem 12 cannot be dropped. We conclude that our main result cannot be extended to general 3-connected matroids.

A 3-connected graph G is *weakly 3-connected* if, for every single edge e of G, at most one of G/e and $G\backslash e$ is 3-connected. Wagner [12] proved the following nice theorem.

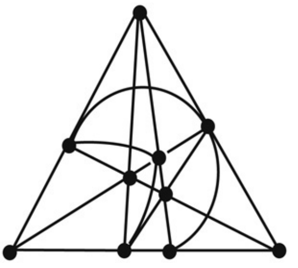

Fig. 4. A counterexample H_{10}.

Theorem 16. *Let G be a weakly* 3*-connected graph on at least four vertices. Then either,*

1. *G has a triad T^* and $e \in T^*$ such that $\lfloor G\backslash e \rfloor$ is weakly* 3*-connected, or*
2. *G has a triangle T and $e \in T$ such that $\lceil G/e \rceil$ is weakly* 3*-connected,or*
3. *G has a degree-three vertex u such that $G\backslash u$ is weakly* 3*-connected, or*
4. *G has a triangle T such that G/T is weakly* 3*-connected, or*
5. *$G = K_4$.*

In the same paper [12], Wagner also conjectured that this result could be extended to 3-connected matroids. Here deleting a vertex of degree three in the above theorem will be replaced by deleting a vertex-triad in the matroid.

H_{10} is also a counterexample to Wagner's conjecture. First, as each element is in a triangle, H_{10} is indeed weakly 3-connected. Since H_{10} does not have any triad, the first or the third statement of the theorem does not hold. For any element $e \in E(H_{10})$, we have that $\lceil H_{10}/e \rceil = U_{2,4}$, which is not weakly 3-connected. The contraction of any triangle T from H_{10} is isomorphic to $U_{1,7}$, or $U_{1,6} \oplus U_{0,1}$, neither of which is weakly 3-connected. Therefore, an extension of Theorem 16 would fail for this matroid.

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